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EFFECTS OF THE INDUCED MAGNETIC FIELD
ON THE PLASMA COLUMN

by



DAVID SEALE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

AUGUST, 1967



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UNIVERSITY OF ALBERTA
FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled Effects of the Induced Magnetic Field on the Plasma Column, submitted by David Seale, in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

Beginning with Euler's equation of hydrodynamics and the equation of continuity, using the method set forth by Johnson in 1962, the steady state profile and the solution of the helical mode are found for the plasma column in parallel electric and magnetic fields. In this calculation, linear terms in the induced azimuthal magnetic field are retained. In previous work, this induced field was neglected altogether.

In the steady state, the well known pinching due to the azimuthal field is observed. In the helical solution the parameters of the helix are calculated as a function of the critical longitudinal magnetic field at which the helical state is stable and it is found that inclusion of the azimuthal field terms results in a deviation from Johnson's theory for small values of the critical longitudinal magnetic field, when the latter does not completely overshadow the azimuthal field.

Experimental observation of this deviation would require experiments especially designed for this purpose. Under normal conditions of current and density, the effect of the azimuthal field is negligible. To date no such experiment has been performed.

ACKNOWLEDGEMENTS

The writer expresses his gratitude to Dr. B.V. Paranjape who supervised this project and without whose encouragement and help this thesis would not be possible. The writer also expresses his appreciation to Dr. M. Razavy for his advice concerning the writing of the thesis.

The writer gratefully acknowledges the financial support of the National Research Council.

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NOTATION

n	ion-electron density
$\vec{\Gamma}$	flux
T	temperature
τ	frictional relaxation time
ξ	ion-electron creation rate
m	particle mass
$\vec{B}$	magnetic field
$\vec{E}$	electric field
v_z	$= v_{z+} - v_{z-} =$ the sum of the absolute values of the longitudinal velocities. v_z is positive for positive electric field and is proportional to the current.
$\bar{n}$	average ion-electron density
e	absolute value of the electronic charge
c	velocity of light
μ	mobility

$\gamma_0, \gamma_1, \dots$ successive zeros of J_0

$K_0, K_1, \dots$ successive zeros of J_1

R radius of glass tube

D coefficient of diffusion

A, Φ, ψ parameters associated with the azimuthal
magnetic field

ω frequency of the helix

K wave number of the helix

+ and - refer to ions and electrons. The cgs-esu system
of units is used throughout.

Chapter 1: Introduction

1.1 Description of the system

The system to be studied consists of a plasma, that is a partially ionized gas, in parallel electric and magnetic fields. The plasma is contained in a cylindrical glass tube whose diameter is of the order of a few centimeters and whose length is of the order of a meter. The effect of the ends is neglected. The electric field is produced by electrodes at the ends of the tube and the magnetic field by a coil wound around the tube. The plasma state is obtained in a sustained electric discharge in the tube. This discharge is produced by the longitudinal electric field which also serves to vary the current in the discharge. The gas pressure is of the order of 1 mm Hg and the gas density is approximately 10^{18} molecules/cm³. In the range of electric fields used, the charged particle density is of the order of 10^{15} particles/cm³.

The mobilities of the ions and electrons is considered to be determined by collisions with neutrals only since neutrals outnumber charged particles by several orders of magnitude. Since the neutral gas pressure is uniform throughout the tube, the ionic and electronic mobilities are regarded as constant with respect to all variables. These mobilities characterize the gas under consideration in the calculation. As an example, the mobilities of ionized helium

are used for the numerical work.

The walls of the tube act as an ion-electron pair sink. This is reflected in the boundary condition that the charged particle density at the walls is zero. This results in a negative radial gradient of charge density. This leads to an outward radial diffusion of ions and electrons whose interaction with the longitudinal magnetic field produces an azimuthal force on the particles. The resulting azimuthal motion and the longitudinal motion of the charged particles, due to the external electric field, combine to produce a helical motion. This helical state is studied in chapter 3.

To avoid a buildup of charges, the radial fluxes of ions and electrons must be equal. Due to the difference in the mobilities of ions and electrons, we must have an internal radial electric field to draw ions to the walls and retard the electrons. The densities of ions and electrons are assumed equal at every point. The possibility of an internal electric field, however, is not excluded. These are the assumptions of quasi-neutrality, which is justified in chapter 2.

The neutral gas remains at approximately room temperature. However the temperature of the electrons is several orders of magnitude greater in the range of pressures and electric field used. Due to the infrequency of collisions between electrons and neutrals, the electrons acquire much energy from the electric field before transferring energy to the gas in a collision. The average energy so gained must

equal the average loss at each collision. Since most collisions are elastic, the transferred energy is a small fraction of the total energy (Howatson 1965). Therefore the average energy of the electrons is great in an electric field and consequently their temperature is high. The ions, on the other hand, experience frequent collisions and exchange appreciable kinetic energy with neutrals. Therefore their temperature remains close to the temperature of the gas. Since the charged particle temperatures are determined by the frequency of collisions with neutrals and since the density of neutrals is constant in space, we assume the charged particle temperatures independent of spatial coordinates.

Due to the sink at the walls, a concentration of charges is observed in the axial region. This effect corresponds to the steady state solution given in chapter 2. As the externally applied longitudinal magnetic field is increased, a screw-type state is observed. This helical state is observed to be unstable for sufficiently large magnetic fields. The mathematical description of this helical state is given in chapter 3.

The variables of this problem are the density of charged particles, the magnetic field and the electric field. The longitudinal electric and magnetic fields, since they are externally applied, are treated as independent variables. The internal radial electric field, the density, and the induced magnetic fields are the dependent variables.

1.2 Previous work in this field

The existence of the critical magnetic field was first described by Guthrie and Wakerling in 1949. The first satisfactory theoretical treatment of the problem was performed by Kadomtsev and Nedospasov in 1960. Their calculation was refined by Johnson and Jerde in 1962 and their treatment remains the most complete on the subject to date.

The theory of Kadomtsev and Nedospasov begins with the equation of continuity and the equations of motion, and by dictating a perturbed radial dependence for the electron density and potential, using the zeroth order expressions, obtains a dispersion relation. This is done by a normal mode analysis.

Johnson generalizes this method by removing the restriction of having to specify the radial dependences of density and potential, although in his actual calculations, he does not go beyond the zeroth order. The advantage of his treatment is that it could be easily be adapted to include higher orders.

Furthermore, Kadomtsev and Nedospasov neglect ionic diffusion and interaction with the magnetic field, thereby limiting the application of the theory to cases of large electron-ion mobility ratios. Johnson, however, makes no such approximations, and his calculation remains symmetric with respect to ions and electrons.

None of the theoretical work in this field has ever included the term due to the internal magnetic field in the

azimuthal direction. The explanation for this is that first, the longitudinal magnetic field normally overshadows the azimuthal field and second, inclusion of this term complicates the algebra enormously.

As a consequence of the omission of this term we can expect that the results of the above authors would be unrealistic for small critical magnetic fields when the azimuthal field becomes comparable to the longitudinal field.

1.3 The improvement to be made

The object of the present calculation is to include the term due to the induced azimuthal magnetic field. Johnson's method is extended to include this term and, as it will be shown later, the results for small critical magnetic fields are rendered physically realistic.

The additional term is assumed to be small, and only the linear terms are kept. The generality of Johnson's work is forsaken in taking only zeroth order terms from the start because of the complexity of the algebra. However, since Johnson has carried out numerical calculations only to the zeroth order anyway, the results of this calculation reduce to his if the parameter associated with the azimuthal magnetic field is reduced to zero. The difference is that the present calculation could not easily be adapted to include higher orders.

In the next chapter, orders of magnitude are given

for the quantities involved and it is found that the effect of the additional term is not negligible under certain circumstances. The additional term is most important in cases of large current and of large average density. Since Johnson's theory requires no upper bound on these quantities, there is a region where the results of this calculation may be observed. The system may be placed in conditions where the effect of the azimuthal field may be observed without sacrificing the validity of Johnson's basic theory.

1.4 Outline of the method

The treatment is divided into two parts. First the steady state is studied. The steady state is defined by the condition that the density at every point in space remains constant in time. Beginning with the equations of motion, which are derived from Euler's hydrodynamic equation, and the continuity equation, a solution in the form of a series of Bessel functions of order zero is obtained. The coefficients of this series, except for the first, are zero for zero azimuthal magnetic field, yielding Johnson's solution, which is a single zero-order Bessel function. The coefficient of the first order term is calculated.

In the second part, beginning with the same equations, the helical mode is studied. A solution in the form of a series of Bessel functions of order one is substituted

and from the relation between the coefficients, using a stability criterion, an equation containing the wavelength of the helix, the critical longitudinal magnetic field, and the longitudinal electric field is obtained. The equation also contains the azimuthal magnetic field as a parameter and reduces to Johnson's corresponding equation if the azimuthal field is reduced to zero.

Chapter 2: Basic Equations and steady state solution

2.1 Introduction of the basic equations

We obtain the equations of motion of the system from Euler's equation of hydrodynamics. Euler's equation may be written

$$\rho \left\{ \frac{\partial}{\partial t} + (\vec{v} \cdot \text{grad}) \right\} \vec{v} + \nabla p = \vec{F}$$

where ρ is the mass density, $\vec{v}$ the velocity, p the pressure and $\vec{F}$ the external forces (Sommerfeld). In the further use of this equation, we neglect $(\vec{v} \cdot \text{grad})\vec{v}$ in comparison with $\text{grad } p$ or the external forces.

We treat the plasma as two fluids and write an Euler equation for each of the fluids, electronic and ionic. We may write these equations

$$n_{\pm} m_{\pm} \frac{\partial \vec{v}_{\pm}}{\partial t} + \nabla p_{\pm} = \vec{F}_{\pm}$$

where $n_{\pm}$ is the particle density and $m_{\pm}$ the particle mass. The + and - indices refer to ions and electrons.

The external forces acting on these fluids are the frictional force and the forces due to the electric and magnetic fields. These may be written

$$F_{\text{friction}\pm} = - \frac{n_{\pm} m_{\pm} \vec{v}_{\pm}}{\tau_{\pm}}$$

$$\vec{F}_{\text{electric}\pm} = \pm n_{\pm} e \vec{E}$$

$$\vec{F}_{\text{magnetic}\pm} = \pm n_{\pm} e \vec{v}_{\pm} \times \vec{B}$$

where $\tau_{\pm}$ is the relaxation time responsible for the frictional force, $\vec{E}$ the electric field, $\vec{B}$ the magnetic field, and e the absolute value of the electronic charge. Though the magnetic force is quadratic in the longitudinal velocity component v_z , this term is larger than the term involving $\vec{v} \cdot (\text{grad } \vec{v})$ since the latter must involve either the radial or azimuthal velocity which is much smaller than v_z .

We note that the azimuthal magnetic field interacts with the longitudinal current to create a radial force. Therefore the azimuthal field will modify the radial motion whose interaction with the longitudinal magnetic field results in the azimuthal motion, as seen in chapter 1. Consequently, the azimuthal field will affect the helix.

To solve the problem, we make the assumption of quasi-neutrality. The densities of ions and electrons are regarded as equal everywhere without excluding the possibility of an internal electric field. This assumption is based on the fact that departures from neutrality, insignificant from the point of view of density, create strong electric fields. This assumption permits the removal of the + and - indices from n .

We assume both ionic and electronic gases ideal and write the equation of state as

$$p_{\pm} = n kT_{\pm}$$

A boundary condition of the problem is dictated by the sink at the walls. This is written as

$$n(R) = 0$$

where R is the radius of the tube.

The other hydrodynamic equation which we must use is the continuity equation for a gas whose particles are not conserved, which may be written as

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = n\xi$$

where ξ is the number of pairs created per electron per unit time. Due to the dominance of recombinations at the boundary, we neglect recombination and assume only regeneration in the volume of the tube. The rate of production of ion-electron pairs is due to ionization of the neutral molecules by collision with electrons and is directly proportional to the number of electrons. Therefore, the rate of regeneration/cm³ is $n\xi$ where n is the number of electrons.

Since the frequency of the helix is much less than $1/\tau$, we neglect the time derivative of the velocity in Euler's

equation (Spitzer, Johnson). This assumption is equivalent to assuming the system in equilibrium with the external forces at every instant.

The basic equations may then be written

$$\vec{\Gamma}_{\pm} = \mu_{\pm} \left(-\frac{kT_{\pm}}{e} \nabla n \pm n \vec{E} \pm \vec{\Gamma}_{\pm} \times \vec{B} \right) \quad (1)$$

$$\frac{\partial n}{\partial t} + \nabla \cdot \vec{\Gamma}_{\pm} = n \xi \quad (2)$$

$$p_{\pm} = n kT_{\pm} \quad (3)$$

where we define $\vec{\Gamma}_{\pm}$ and $\mu_{\pm}$ as follows

$$\vec{\Gamma}_{\pm} = n \vec{v}_{\pm}$$

$$\mu_{\pm} = \frac{e\tau_{\pm}}{m_{\pm}}$$

We note that the continuity equation requires that

$$\nabla \cdot \vec{\Gamma}_{+} = \nabla \cdot \vec{\Gamma}_{-}$$

2.2 Steady state solution

We define the steady state by the following condition

$$\frac{\partial n}{\partial t} = 0$$

Writing the r components of the equations of motion, we obtain

$$\begin{aligned} -\mu_- \left[-n E_{or} - n (v_{-\phi} B_{oz} - v_{-z} B_{o\phi}) - \frac{kT_-}{e} \frac{\partial n}{\partial r} \right] &= n v_{-r} \\ +\mu_+ \left[n E_{or} + n (v_{+\phi} B_{oz} - v_{+z} B_{o\phi}) - \frac{kT_+}{e} \frac{\partial n}{\partial r} \right] &= n v_{+r} \end{aligned} \quad (4)$$

which are first order, partial differential equations in n . One of the boundary conditions for equations (4) is dictated by the sink at the surface. That is

$$n(R) = 0$$

Because of the cylindrical symmetry in the steady state, to avoid charge separation, we must have

$$v_{-r} = v_{+r} = v_r$$

Solving for $v_{+\phi}$ and $v_{-\phi}$ from the ϕ component of equation (2) we get

$$\begin{aligned} v_{+\phi} &= - \mu_+ v_r B_{oz} \\ v_{-\phi} &= \mu_- v_r B_{oz} \end{aligned} \quad (5)$$

since $E_{o\phi}$ and $\frac{\partial n}{\partial \phi}$ must be zero for cylindrical symmetry and B_{or} must be zero since it must satisfy Maxwell's equation

$$\nabla \cdot \vec{B} = 0$$

and also be cylindrically symmetric.

We use the lowest term in the power series expansion of $B_{o\phi}$, that is the term linear in r . We assume

$$B_{o\phi} \approx \frac{\bar{n} e v_z r}{c^2} \quad (6)$$

where $\bar{n}$ is the average ion-electron density and

$$v_z = v_{z+} - v_{z-}$$

which is proportional to the current.

Between equations (4) we eliminate E_{or} . This yields, using substitutions (5) and (6), and solving for Γ_r

$$\Gamma_r = -\mu'' \left[\frac{n v_z^2 \bar{n} e r}{c^2} - \frac{2 kT}{e} \frac{\partial n}{\partial r} \right] \quad (7)$$

where

$$\mu'' = \frac{1}{(\mu_+ + \mu_-) \left(\frac{1}{\mu_+ \mu_-} + B_{oz}^2 \right)}$$

and

$$T = \frac{T_+ + T_-}{2} = \text{average temperature}$$

Substituting the expression for Γ_r from equation (7) into the continuity equation (2) we obtain the second order differential equation

$$\frac{\partial^2 n}{\partial r^2} + \left[\frac{1}{r} + r \left(\frac{v_z^2 \bar{n} e^2}{2c^2 kT} \right) \right] \frac{\partial n}{\partial r} + \left[\frac{v_z^2 \bar{n} e^2}{c^2 kT} + \frac{e \xi}{2\mu kT} \right] n = 0$$

Since n depends only on r we may replace the partial derivatives by total derivatives.

We define the dimensionless variables ρ and N as follows

$$\rho = r/R \quad N = n/\bar{n}$$

where R is the radius of the tube and $\bar{n}$ is the average electron-ion density. The differential equation may then be written

$$\frac{d^2 N}{d\rho^2} + \left(\frac{1}{\rho} + \frac{A\rho}{2} \right) \frac{dN}{d\rho} + (A + B) N = 0 \quad (8)$$

where

$$A = \frac{v_z^2 \bar{n} e^2 R}{c^2 kT} \quad B = \frac{e R^2 \xi}{2\mu kT}$$

and $N(R) = 0$

We note that A and B are dimensionless.

If we put A equal to zero, which corresponds to putting $B_{0\phi}$ to zero, equation (8) reduces to an exact Bessel equation, whose solution is a Bessel function of order

zero and whose first zero occurs at R , which corresponds to Johnson's solution for the steady state. We can examine the effect of $B_{0\phi}$ by writing the solution as a series of zero order Bessel functions whose first, second, etc., zeros occur at $r = R$.

The solution is written as

$$N = N_0 [J_0(\gamma_0 \rho) + a_1 J_0(\gamma_1 \rho) + \dots]$$

where $\gamma_0, \gamma_1, \dots$ are the successive zeros of J_0 .

Substituting this solution into equation (8), multiplying by $J_0(\gamma_1 \rho) \rho d\rho$ and integrating over ρ from zero to one, we obtain an expression for a_1 whose linear term in A is

$$\frac{A \int_0^1 \rho^2 J_0'(\gamma_0 \rho) J_0(\gamma_1 \rho) d\rho}{2 (\gamma_1^2 - \gamma_0^2) \int_0^1 J_0^2(\gamma_1 \rho) \rho d\rho}$$

The numerical values of the integrals give

$$a_1 = 0.0346 A + \dots$$

Therefore the lowest order expression for a_1 becomes

$$a_1 \approx 0.0346 \frac{v_z^2 \bar{n} e^2 R^2}{c^2 kT}$$

and the solution may be written

$$N \approx N_0 \left[J_0(\gamma_0 \rho) + 0.0346 \frac{v_z^2 \bar{n} e^2 R^2}{c^2 kT} J_0(\gamma_1 \rho) \right] \quad (9)$$

The orders of magnitude of the quantities in the above solution are

$$v_z \sim 10^6 \text{ cm/sec}$$

$$\bar{n} \sim 10^{15} / \text{cm}^3$$

$$e \sim 10^{-10} \text{ esu}$$

$$R \sim 1 \text{ cm}$$

$$c \sim 10^{10} \text{ cm/sec}$$

$$k \sim 10^{-16} \text{ ergs/degree K}$$

$$T \sim 10^4 \text{ degrees K} \quad (\text{Brown 1959})$$

which give an order of magnitude of 10^{-1} for A . For the approximation of taking only the linear term in the expansion for N , we must have $a_1 \ll 1$. That is we must have $A \ll 30$. To plot the profile so that the effect of the azimuthal field is not negligible, we take $A = 1$. In this case the solution is

$$N \approx N_0 \left[J_0(\gamma_0 \rho) + 0.0346 J_0(\gamma_1 \rho) \right] \quad (10)$$

where the numerical values of the γ 's are

$$\gamma_0 = 2.405$$

$$\gamma_1 = 5.520$$

For this example the results of the calculation are shown in figure 1. From the graphs we observe that the calculated result is reasonable since it is known that the effect of $B_{0\phi}$ in semiconductors is to pinch the column.

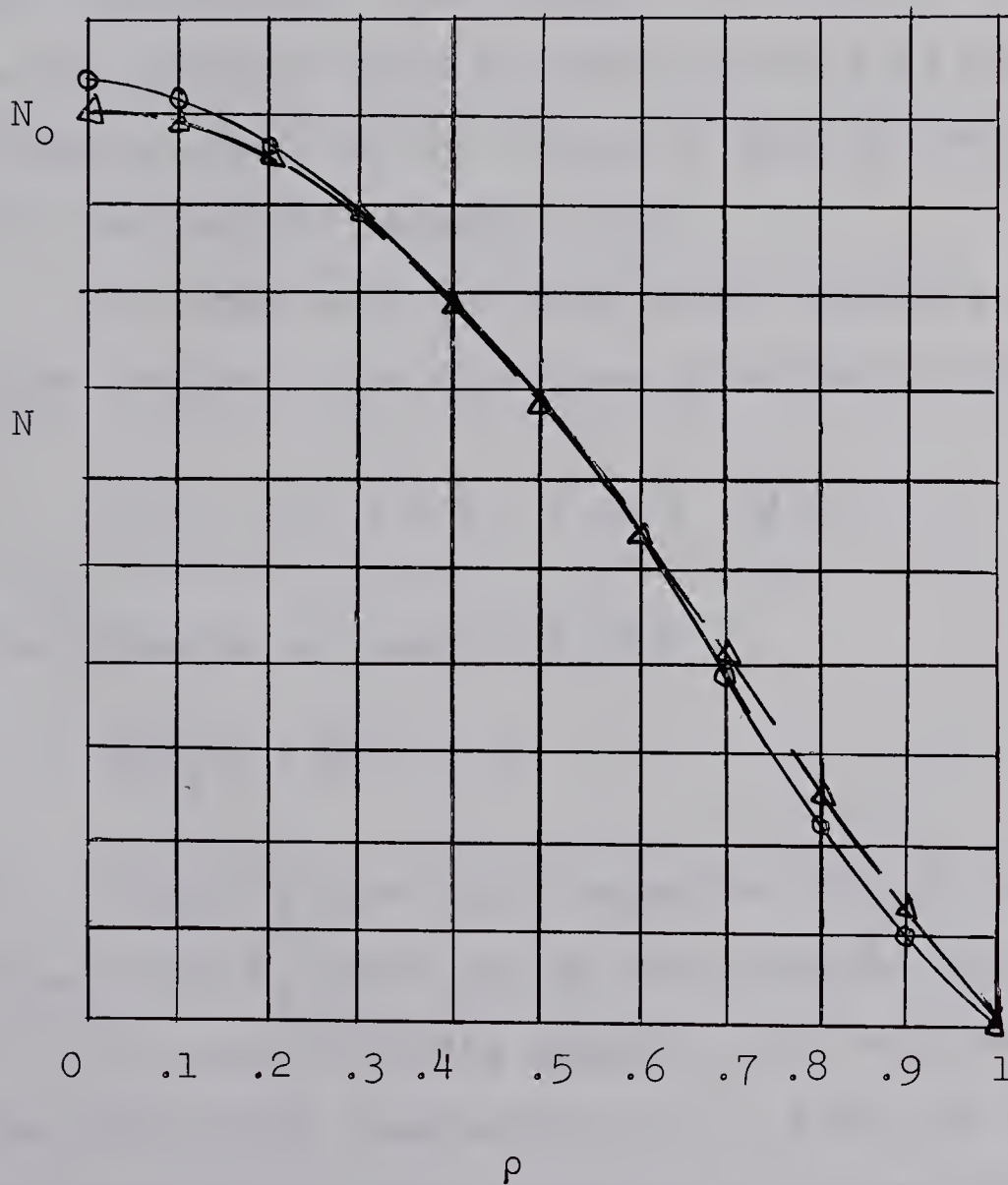


figure 1

Graph of equation 10 (solid curve) for parameter $A = 1$, and Johnson's solution, $J(\gamma_0 \rho)$ (dashed curve)

Chapter 3: Helical solution

It has been established both theoretically and experimentally that a helical mode exists for this system. Johnson has developed the theory and performed calculations for the helical mode. In this section, we shall examine the effect produced by the terms due to $B_{0\phi}$ which Johnson has neglected.

For longitudinal magnetic fields below a critical value, which we shall calculate, the helical mode decays. Above the critical field the helical state is unstable. The imaginary part of the frequency depends critically on the longitudinal magnetic field.

We begin with the same basic equations as in the previous chapter. The equations of motion are as before

$$\vec{\Gamma}_{\pm} = \mu_{\pm} [\pm n \vec{E} \pm \vec{\Gamma}_{\pm} \times \vec{B} - \nabla(kT_{\pm} n)] \quad (10)$$

and the equation of continuity is

$$\nabla \cdot \vec{\Gamma}_{\pm} = - \frac{\partial n}{\partial t} + n\xi \quad (11)$$

First we must solve equation (10) for an explicit expression for $\vec{\Gamma}_{\pm}$ which may be substituted into equation (11). To do this we write equation (10) as three equations in the cylindrical components of $\vec{\Gamma}_{\pm}$. From the expressions of its components we can construct a vectorial expression for $\vec{\Gamma}_{\pm}$. This expression may be written

$$\vec{\Gamma}_{\pm} = \frac{\mu_{\pm}^3 n \vec{B} \cdot \vec{E} \vec{B} - \mu_{\pm}^3 \frac{kT_{\pm}}{e} \vec{B} \vec{B} \cdot \nabla n \pm n [\vec{E} \mp \vec{B} \times \vec{E} \mu_{\pm}]}{1 + \mu_{\pm}^2 B^2} + \frac{\frac{\mu_{\pm} kT_{\pm}}{e} [\mu_{\pm} \vec{B} \times \nabla n \mp \nabla n]}{1 + \mu_{\pm}^2 B^2} \quad (12)$$

Combining equations (11) and (12) to eliminate $\vec{\Gamma}_{\pm}$ we obtain

$$-\frac{\partial n}{\partial t} + n\xi = \nabla \cdot \left\{ \left[\pm \mu_{\pm}^3 n \vec{B} \cdot \vec{E} \vec{B} - \frac{\mu_{\pm}^3 kT_{\pm}}{e} \vec{B} \vec{B} \cdot \nabla n \pm n(\vec{E} \mp \mu_{\pm} \vec{B} \times \vec{E}) \pm \frac{\mu_{\pm} kT_{\pm}}{e} (\mu_{\pm} \vec{B} \times \nabla n \mp \nabla n) \right] / (1 + \mu_{\pm}^2 B^2) \right\} \quad (13)$$

This equation may be simplified by using Maxwell's equation

$$\nabla \cdot \vec{B} = 0$$

and the identity from vectorial analysis

$$\nabla \times \nabla n = 0$$

Since the azimuthal field is normally much smaller than the external longitudinal field, we replace $1 + \mu_{\pm}^2 B^2$ by

$1 + \mu_{\pm}^2 B_{OZ}^2$. Using these conditions, we obtain from equation (13)

$$\begin{aligned} (1 + \mu_{\pm}^2 B_{OZ}^2) \left(-\frac{\partial n}{\partial t} + n\xi \right) = & \pm \mu_{\pm}^3 \left[\vec{B} \cdot \nabla n \vec{B} \cdot \vec{E} + \vec{B} \cdot \nabla (\vec{B} \cdot \vec{E}) \right] \\ & - \mu_{\pm}^3 kT_{\pm} \vec{B} \cdot \nabla (\vec{B} \cdot \nabla n) \pm \mu_{\pm} \vec{E} \cdot \nabla n - \mu_{\pm}^2 \left[n \vec{E} \cdot (\nabla \times \vec{B}) - n \vec{B} \cdot (\nabla \times \vec{E}) \right. \\ & \left. + (\vec{B} \times \vec{E}) \cdot \nabla n \right] \pm \mu_{\pm}^2 kT_{\pm} \nabla n \cdot (\nabla \times \vec{B}) - \mu_{\pm} kT_{\pm} \nabla^2 n \end{aligned}$$

To treat this problem, we define a new variable n_1 as follows

$$n_1 = n - n_O$$

where n_O is defined as

$$n_O = N \bar{n}$$

where $\bar{n}$ is the average density and N is the steady state solution given in section 2.2. We assume that n_1 is small and depends on the coordinates and time. Also we must put

$$\vec{E} = \vec{E}_O + \vec{E}_1$$

where $\vec{E}_O$ is the electric field in the steady state and $\vec{E}_1$ is time and coordinate dependent and assumed small. Replacing n and $\vec{E}$ with the above expressions in (14), from which has been subtracted the steady state equation, we obtain

$$\begin{aligned} (1 + \mu_{\pm}^2 B_{OZ}^2) \left(-\frac{\partial n_1}{\partial t} + n_1 \xi \right) = & \pm \mu_{\pm}^3 n_O B_{OZ}^2 \frac{\partial E_{1Z}}{\partial z} \pm \mu_{\pm}^3 B_{OZ}^2 \cdot \\ E_{OZ} \frac{\partial n_1}{\partial z} \pm \mu_{\pm}^3 n_O B_{OZ} E_{OZ} \frac{1}{r} \frac{\partial}{\partial r} (r B_{O\phi}) - \mu_{\pm}^3 \frac{kT_{\pm}}{e} B_{OZ}^2 \left(\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial n_1}{\partial z}) \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r} \frac{\partial^2 n_1}{\partial \varphi \partial z} + \frac{\partial^2 n_1}{\partial z^2} \Big) \pm \mu_{\pm} n_0 \left[\frac{1}{r} \frac{\partial}{\partial r} (r E_{1r}) + \frac{1}{r} \frac{\partial E_{1\varphi}}{\partial \varphi} + \frac{\partial E_{1z}}{\partial z} \right] \\
& - \mu_{\pm}^2 n_0 \left[- \frac{B_{0z}}{r} \frac{\partial (r E_{1\varphi})}{\partial r} + B_{0z} \frac{1}{r} \frac{\partial E_{1r}}{\partial \varphi} \right] \pm \mu_{\pm} \left[\frac{1}{r} \frac{\partial}{\partial r} (r n_1 E_{0r}) \right. \\
& + \left. \frac{\partial n_1}{\partial z} E_{0z} \right] - \frac{1}{r} \frac{\partial (r B_{0\varphi})}{\partial r} E_{0z} \mu_{\pm}^2 n_0 - \mu_{\pm} \left(\frac{1}{r} \frac{\partial n_1}{\partial \varphi} B_{0z} E_{0r} \mu_{\pm} \right) \\
& \pm \frac{\mu_{\pm} k T_{\pm}}{e} \left[- \mu_{\pm} \frac{1}{r} \frac{\partial^2 n_1}{\partial r \partial \varphi} B_{0z} + \mu_{\pm} \frac{1}{r^2} \frac{\partial^2 (r n_1)}{\partial r \partial \varphi} B_{0z} - \nabla^2 n_1 \frac{\mu_{\pm} k T_{\pm}}{e} \right]
\end{aligned}
\tag{15}$$

We now assume a helical form for the density n_1 and the potential V_1 of the field $\vec{E}_1$. This is expressed as

$$n_1 = f(r) \exp [i(Kz + m\varphi + \omega t)]$$

$$V_1 = g(r) \exp [i(Kz + m\varphi + \omega t)]$$

Since we wish to study the simple helix, we set $m = 1$.

The electric field may be expressed in terms of the potential V_1 according to the following set of equations:

$$\begin{aligned}
E_{1r} &= - \frac{\partial g}{\partial r} \frac{V_1}{g} & E_{1\varphi} &= - i \frac{V_1}{r} & E_{1z} &= - i K V_1 \\
\frac{\partial E_{1r}}{\partial r} &= - \frac{\partial^2 g}{\partial r^2} & \frac{\partial (r E_{1\varphi})}{\partial r} &= - i \frac{\partial g}{\partial r} & \frac{\partial E_{1z}}{\partial r} &= - i K \frac{\partial g}{\partial r} \\
\frac{1}{r} \frac{\partial E_{1r}}{\partial \varphi} &= - \frac{i}{r} \frac{\partial g}{\partial r} & \frac{1}{r} \frac{\partial E_{1\varphi}}{\partial \varphi} &= \frac{g}{r^2} & \frac{\partial E_{1z}}{\partial \varphi} &= K g \\
\frac{\partial E_{1r}}{\partial z} &= - \frac{\partial g}{\partial r} i K & \frac{\partial E_{1\varphi}}{\partial z} &= \frac{K g}{r} & \frac{\partial E_{1z}}{\partial z} &= K^2 g
\end{aligned}$$

Defining the quantities $\mu_{\pm}'$ and $D_{\pm}'$ as follows:

$$\mu_{\pm}' = \frac{\mu_{\pm}}{1 + \mu_{\pm}^2 B_{OZ}^2}$$

$$D_{\pm}' = \frac{D_{\pm}}{1 + \mu_{\pm}^2 B_{OZ}^2} = \frac{\mu_{\pm} kT_{\pm}}{1 + \mu_{\pm}^2 B_{OZ}^2}$$

where $D_{\pm}$ is the coefficient of diffusion (see Einstein's relation between the mobility and the diffusion coefficient), replacing $B_{O\varphi}$ by its lowest order expression, that is the term linear in r

$$B_{O\varphi} \approx \frac{\bar{n} v_z e r}{c^2}$$

and defining

$$\Phi = \frac{B_{O\varphi}}{r} = \frac{\bar{n} v_z e}{c^2}$$

from (15) we obtain

$$\begin{aligned} D_{\pm}' \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \mp \mu_{\pm}' \frac{1}{r} \frac{d}{dr} (r E_{Or} f) - [(i\omega - \xi) \pm \mu_{\pm}' F_{OZ} iK + D_{\pm}' K^2 \\ + D_{\pm}'/r^2 - \frac{B_{OZ}}{r} iE_{Or} \mu_{\pm}' \mu_{\pm}'] f + [r F_{OZ} \frac{df}{dr} \mu_{\pm}' \mu_{\pm} \mp \mu_{\pm}' \mu_{\pm}^2 i B_{OZ} E_{OZ} f \\ - 2KB_{OZ} \mu_{\pm}'^2 D_{\pm}' f + 2(E_{OZ} \mu_{\pm} \mp D_{\pm}' iK) \mu_{\pm}' f - iKr E_{Or} \mu_{\pm}' \mu_{\pm} f] \Phi = \\ \mp \mu_{\pm}' \frac{1}{r} \frac{d}{dr} (r n_0 \frac{dg}{dr}) + [\pm \mu_{\pm}' \frac{n_0}{r^2} \pm \mu_{\pm}' n_0 K^2 - \mu_{\pm}' \mu_{\pm}' i \left(\frac{dn_0}{dr} \right) \frac{B_{OZ}}{r}] g \\ + [\mu_{\pm}' \mu_{\pm}'^2 2n_0 B_{OZ} K + \mu_{\pm}' \mu_{\pm}' n_0 iK^2 + \mu_{\pm}' \mu_{\pm}' i \left(\frac{dn_0}{dr} \right) Kr] g \Phi \quad (17) \end{aligned}$$

Since all variables depend only on r , we have replaced the partial derivatives by total derivatives.

In equation (17) terms linear in Φ have been kept whereas terms containing higher powers of Φ have been neglected. We now define a new variable ℓ such that

$$g(r) = \frac{\ell(r)}{n_o(r)} \quad (18)$$

Since $g(R)$, which is essentially the amplitude of the potential of the electric field at the boundary, must be finite and since $n_o(R) = 0$ due to the ion-electron sink at the boundary, $\ell(R)$ must be zero. Rewriting equation (17) in terms of ℓ and f we obtain

$$\begin{aligned} D_{\pm}' \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \mp \mu_{\pm}' \frac{1}{r} \frac{d}{dr} (r E_{or} f) - [(i\omega - \xi) \pm \mu_{\pm}' F_{oz} iK \\ + D_{\pm}' K^2 + D_{\pm}'/r^2 - \frac{B_{oz}}{r} iE_{or} \mu_{\pm}' \mu_{\pm}'] f + [r E_{oz} \frac{df}{dr} \mu_{\pm}' \mu_{\pm}'] \mp \\ \mu_{\pm}' \mu_{\pm}^2 iB_{oz} E_{oz} f - 2KB_{oz} \mu_{\pm}' D_{\pm}' f + 2(E_{oz} \mu_{\pm}' \mp D_{\pm}' iK) \mu_{\pm}' f - \\ KriE_{or} \mu_{\pm}' \mu_{\pm}' f] \Phi = \mp \mu_{\pm}' \frac{1}{r} \frac{d}{dr} \left(r \frac{d\ell}{dr} \right) \pm \mu_{\pm}' \frac{1}{r} \frac{d(\ell r p)}{dr} \\ + [\pm \frac{\mu_{\pm}'}{r^2} \pm \mu_{\pm}' K^2 - \mu_{\pm}' \mu_{\pm}' \frac{B_{oz}}{r}] \ell + [\pm \mu_{\pm}' \mu_{\pm}^2 2B_{oz} K + 2\mu_{\pm}' \mu_{\pm}' iK \\ + \mu_{\pm}' \mu_{\pm}' ipKr] \Phi \ell \end{aligned}$$

$$\text{where } p(r) = \frac{1}{n_o} \frac{dn_o}{dr} \quad (20)$$

We know from Johnson's work that the approximate solution to this problem without the terms in Φ , is a Bes-

sel function of order one, whose first zero occurs at the boundary. Since we can vary the parameter Φ over a wide range without losing the validity of Johnson's theory, we note that the introduction of Φ is consistent with Johnson's approximations. We may write the solution to the problem dealing with the terms in Φ as the sum of first order Bessel functions whose successive zeros occur at $r = R$. We write

$$\begin{aligned} f &= f_0 J_1(K_0 r) + f_1 J_1(K_1 r) + \dots \\ \ell &= \ell_0 J_1(K_0 r) + \ell_1 J_1(K_1 r) + \dots \end{aligned} \quad (21)$$

where $K_0, K_1, \dots$ are the first, second, ... zeros of J_1 . Substituting expressions (21) into equation (19), multiplying by

$$\frac{J_1(K_0 r) r dr}{D_{\pm}^2}$$

and integrating over r from $r = 0$ to $r = R$, we obtain two equations relating f_0 and ℓ_0 with terms in $f_1, \dots$ and $\ell_1, \dots$. All terms containing $f_1, \dots$ and $\ell_1, \dots$ are multiplied by integrals of the form

$$\int_0^R J_1(K_0 r) J_1(K_1 r) r dr, \quad \int_0^R J_1(K_0 r) J_1(K_2 r) r dr, \dots$$

If the range of integration were infinite, all these integrals would vanish, and this method would reduce to the standard transform method. However in the case of a fin-

ite range of integration, these integrals are finite but much smaller than the integral

$$\int_0^R J_1^2(K_0 r) r dr$$

which occurs in the coefficients of the f_0 and ℓ_0 terms. This is due to the oscillation of the integrands which contain Bessel functions of different arguments. Also, since $f_1, \dots$ and $\ell_1, \dots$ are only corrections, they must be small compared to f_0 and ℓ_0 . Consequently, we neglect all terms containing $f_1, \dots$ and $\ell_1, \dots$ and are left with two equations in f_0 and ℓ_0 of the form

$$A_+ f_0 + B_+ \ell_0 = 0$$

$$A_- f_0 + B_- \ell_0 = 0$$

where

$$\begin{aligned} A_{\pm} = & K_0^2 kT_{\pm} \pm I_3 \alpha - \frac{\omega_2}{\mu_{\pm}} + \frac{i\omega_1}{\mu_{\pm}} - \frac{\xi}{\mu_{\pm}} \pm \frac{\mu_{\pm}}{\mu_{\pm}} E_{oz} iK + \frac{\mu_{\pm}}{\mu_{\pm}} K^2 kT_{\pm} \\ & - i\mu_{\pm} B_{oz} I_1 \alpha + \Phi[-E_{oz} \mu_{\pm} I_4 \pm \mu_{\pm}^2 B_{oz} E_{oz} i + 2KB_{oz} \mu_{\pm}^2 kT_{\pm} \\ & - 2E_{oz} \mu_{\pm} \pm 2i\mu_{\pm} KkT_{\pm} + i\mu_{\pm} K I_2 \alpha] \end{aligned}$$

and

$$\begin{aligned} B_{\pm} = & \pm [K_0^2 + I_3 + \frac{\mu_{\pm}}{\mu_{\pm}} K^2] - \mu_{\pm} i B_{oz} I_1 + \Phi [\pm 2\mu_{\pm}^2 B_{oz} K \\ & + 2\mu_{\pm} iK + \mu_{\pm} iK I_2] \end{aligned}$$

and where

$$\alpha = \frac{\mu'_+ kT_+ - \mu'_- kT_-}{\mu'_+ + \mu'_-}$$

$$\omega = \omega_1 + i\omega_2$$

ω_1 and ω_2 being real, and

$$\begin{aligned} I_1 &= \frac{2}{R^2 [J_0(K_0 r)]^2} \int_0^R p J_1(K_0 r) dr \\ I_2 &= \frac{2}{R^2 [J_0(K_0 r)]^2} \int_0^R p J_1^2(K_0 r) r^2 dr \\ I_3 &= \frac{2}{R^2 [J_0(K_0 r)]^2} \int_0^R \frac{d}{dr} \left[r \frac{E_{or}}{\alpha} J_1(K_0 r) \right] J_1(K_0 r) dr \\ I_4 &= \frac{2}{R^2 [J_0(K_0 r)]^2} \int_0^R J_1'(K_0 r) J_1(K_0 r) r^2 dr \end{aligned}$$

To have a non-trivial solution for f_0 and l_0 we must set the determinant of the coefficients of equation (22) equal to zero. That is

$$A_+ B_- - A_- B_+ = 0 \quad (24)$$

Dividing equation (24) into its real and imaginary parts, we obtain two equations of the form

$$A_1 \omega_1 + B_1 \omega_2 = C_1 \quad (25)$$

$$A_2 \omega_1 + B_2 \omega_2 = C_2$$

From these equations the imaginary frequency may be written

$$\omega_2 = \frac{\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$$

At the critical field the helical state must neither grow nor decay. That is, at the critical field, $\omega_2 = 0$. Therefore we put

$$A_1 C_2 - A_2 C_1 = 0 \quad (26)$$

recognizing that henceforth B_{oz} represents the critical magnetic field.

From equation (25)

$$\begin{aligned} A_1 &= \left(\frac{\mu_-}{\mu_+} - \frac{\mu_+}{\mu_-} \right) [B_{oz} I_1 - (2 + I_2) K \Phi] \\ A_2 &= \left[\left(\frac{1}{\mu_+} + \frac{1}{\mu_-} \right) (K_o^2 + I_3) + (\mu_- + \mu_+) \frac{K^2}{\mu_+ \mu_-} + \left(\frac{\mu_-^2}{\mu_+} + \frac{\mu_+^2}{\mu_-} \right) 2B_{oz} K \Phi \right] \\ C_1 &= \left[K_o^2 kT_+ + I_3 \alpha - \frac{\xi}{\mu_+} + \frac{\mu_+}{\mu_+} K^2 kT_+ - \mu_+ (E_{oz} I_4 - 2KB_{oz} \mu_+ kT_+ \right. \\ &\quad \left. + 2E_{oz}) \Phi \right] \left[- (K_o^2 + I_3) - \frac{\mu_-}{\mu_-} K^2 - 2\mu_-^2 B_{oz} K \Phi \right] - \\ &\quad \left[\frac{\mu_+}{\mu_+} E_{oz} K - \mu_+ B_{oz} I_1 \alpha + \mu_+ (\mu_+ B_{oz} E_{oz} + 2K kT_+ + K I_2 \alpha) \Phi \right] \\ &\quad \cdot [(2 + I_2) K \mu_- - \mu_- B_{oz} I_1] - \left[K_o^2 kT_- - I_3 \alpha - \frac{\xi}{\mu_-} \right. \\ &\quad \left. + \frac{\mu_-}{\mu_-} K^2 kT_- - \mu_- (E_{oz} I_4 - 2KB_{oz} \mu_- kT_- - \mu_- kT_- + 2E_{oz}) \Phi \right] \cdot \end{aligned}$$

$$\begin{aligned}
& [(K_o^2 + I_3) + \frac{\mu_+}{\mu_+} K^2 + 2\mu_+^2 B_{oz} K\Phi] + [- \frac{\mu_-}{\mu_-} E_{oz} K - \mu_- B_{oz} I_1 \alpha \\
& - \mu_- (\mu_- B_{oz} E_{oz} + kT_- 2K - KI_2 \alpha) \Phi] [- \mu_+ B_{oz} I_1 + \mu_+ (2 + I_2) K\Phi] \\
C_2 = & [K_o^2 kT_+ + I_3 \alpha - \frac{\xi}{\mu_+} + \frac{\mu_+}{\mu_+} K^2 kT_+ + (2KB_{oz} \mu_+ kT_+ - 2E_{oz} \\
& + I_4 E_{oz}) \Phi] [- \mu_- B_{oz} I_1 + \mu_- K (2 + I_2) \Phi] + [\frac{\mu_+}{\mu_+} E_{oz} K \\
& + \mu_+ B_{oz} I_1 \alpha + \mu_+ (B_{oz} E_{oz} + 2K kT_+ + K I_2 \alpha)] \cdot \\
& [- (K_o^2 + I_3) - \frac{\mu_-}{\mu_-} K^2 - 2\mu_-^2 B_{oz} K\Phi] - [K_o^2 kT_- - I_3 - \frac{\xi}{\mu_-} \\
& + \frac{\mu_-}{\mu_-} K^2 kT_- - (E_{oz} I_4 - 2KB_{oz} \mu_- kT_- - 2E_{oz}) \mu_- \Phi] \cdot \\
& [-\mu_+ B_{oz} I_1 + \mu_+ K\Phi (2+I_2)] - [(K_o^2 + I_3) + \frac{\mu_+}{\mu_+} K^2 + \mu_+^2 2B_{oz} K\Phi] \cdot \\
& [- \frac{\mu_-}{\mu_-} E_{oz} K - \mu_- B_{oz} I_1 \alpha - \mu_- (\mu_- B_{oz} E_{oz} + 2K kT_- - K\alpha I_2) \Phi]
\end{aligned}
\tag{27}$$

Replacing A_1, A_2, C_1, C_2 in equation (26), after much algebra we obtain the equation

$$\begin{aligned}
& G_1 z^3 + G_2 z^2 + G_3 z + G_4 + [G_8 z + G_7 z + G_6 z] \psi \\
& = -[a_7^2 (1+y)^2 + 4y] b_{31} [G_{12} (1+C_{11}) z + G_{12} z \\
& + (G_9 z + G_{10} z^2 + G_{11}) \psi]
\end{aligned}
\tag{28}$$

where

$$C_{11} = I_3 / K_o^2$$

$$D_{11} = -I_1/K_0^2$$

$$A_{01} = 1 - \gamma_0^2/K_0^2$$

$$z = (1 + \mu_+ \mu_- B_{0z}^2) K^2/K_0^2$$

$$b_{31} = \frac{e E_{0z}}{K_0 k (T_+ + T_-)}$$

$$y = \mu_+ \mu_- B_{0z}^2$$

$$a_7 = \frac{\mu_-' \mu_- - \mu_+' \mu_+}{\mu_+' + \mu_-'} B_{0z}$$

$$\psi = \frac{\Phi y^{\frac{1}{2}}}{K_0 B_{0z}}$$

and where the coefficients G are

$$G_1 = (1 + a_7^2)^2$$

$$G_2 = 2(1 + C_{11})(1 + a_7^2)(1 + \frac{a_7^2 y}{2}) + A_{01} (1 + a_7^2)^2$$

$$G_3 = (1 + C_{11})^2 (1 + a_7^2 y) + 2A_{01} (1 + C_{11})(1 + a_7^2) \\ + D_{11}^2 a_7^2 (1 + y)$$

$$G_4 = A_{01} [(1 + C_{11})^2 + D_{11} a_7^2]$$

$$G_6 = a_7^2 D_{11} [2(1 + C_{11}) (\frac{1+y}{y})^{\frac{1}{2}} + \frac{I_2 (1 + C_{11})}{y^{\frac{1}{2}} (1+y)^{\frac{1}{2}}} \\ - C_{11} \frac{(2 + I_2)}{(1+y)^{\frac{1}{2}} y^{\frac{1}{2}}} - 2D_{11} (\frac{y}{1+y})^{\frac{1}{2}} - \frac{(2+I_2)}{(1+y)^{\frac{1}{2}} y^{\frac{1}{2}} (\frac{K_0}{K})^2} \\ - D_{11} (\frac{\gamma_0}{K_0})^2] + \frac{(2+I_2)}{(1+y)^{\frac{1}{2}} y^{\frac{1}{2}}} a_7^2 A_{01} D_{11} + (1+C_{11}) \cdot \\ [\frac{2(a_7^2 (1+y)^2 + 2y)}{(1+y)^{\frac{1}{2}} y^{\frac{1}{2}}} + \frac{2(D_{11}+C_{11}) y^{\frac{1}{2}}}{(1+y)^{\frac{1}{2}}}] + [\frac{2y^{\frac{1}{2}}}{(1+y)^{\frac{1}{2}}}]$$

$$\begin{aligned}
& + \frac{2(1+y)^{1/2}}{y^{1/2}} a_7^2 \big] \big[(1+C_{11}) A_{01} \big] \\
G_7 &= 2a_7^2 D_{11} \big[(2+I_2) \left(\frac{1+y}{y} \right)^{1/2} \big] + (1+C_{11}) \cdot \left[\frac{2[a_7^2(1+y)^2 + 2y]}{(1+y)y} \right. \\
& \quad \left. + \frac{4y^{3/2}}{(1+y)^{3/2}} \right] + 2 \left[\frac{y + a_7^2(1+y)}{(1+y)^{1/2} y^{1/2}} \right] \cdot [A_{01}(1+a_7^2) + (1+C_{11}) \cdot \\
& \quad (1+a_7^2 y) + 2(1+a_7^2) \left[\frac{a_7^2(1+y)^2 + 2y}{y^{1/2}} + C_{11} y^{1/2} + D_{11} y^{1/2} \right. \\
& \quad \left. - 2 \left(\frac{Y_0}{K_0} \right)^2 y^{1/2} \right] \frac{1}{(1+y)^{1/2}} \\
G_8 &= \left[\frac{2y^{1/2}}{(1+y)^{1/2}} + \frac{2(1+y)^{1/2}}{y^{1/2}} a_7^2 \right] [1+a_7^2] + (1+a_7^2) \left[\frac{2a_7^2(1+y)^2 + 2y}{(1+y)^{3/2} y^{1/2}} \right. \\
& \quad \left. + \frac{4y^{3/2}}{(1+y)^{3/2}} \right] \\
G_9 &= a_7^2 D_{11} / y^{1/2} + (2+I_2)(1+C_{11}) a_7^2 / y^{1/2} + (1+C_{11}) \left(\frac{I_2 - I_4}{y^{1/2}} \right) \\
& \quad + \left[\frac{2y^{1/2}}{(1+y)^{1/2}} + \frac{2(1+y)^{1/2}}{y^{1/2}} a_7^2 \right] D_{11} (1+y)^{1/2} + (1+a_7^2) \cdot \\
& \quad [-(2+I_4)(1+C_{11}) / y^{1/2} + y^{1/2} D_{11}] \\
G_{10} &= (1+a_7^2) (I_2 - I_4) / y^{1/2} \\
G_{11} &= -a_7 D_{11} \frac{(1+C_{11})}{y^{1/2}} \frac{[a_7^2(1+y)^2 + 2y]}{[a_7^2(1+y)^2 + 4y]^{1/2}} + (1+C_{11}) [y^{1/2} D_{11} \\
& \quad - (2+I_4)(1+C_{11}) / y^{1/2}] \\
G_{12} &= D_{11} (1+y)^{1/2} (1+a_7^2)
\end{aligned}$$

Setting ψ equal to zero in equation (28), we obtain Johnson's corresponding equation.

For the numerical calculations, we shall consider helium for which

$$\frac{\mu_-}{\mu_+} = 43.7$$

We are now left with an algebraic equation in the three variables y , z and b_{31} .

We wish to eliminate states which are stable only for a precise wavelength in favour of those stable in a small region about the stable wavelength. The latter are observed experimentally but the former, corresponding to a perfectly monochromatic state, are not.

To impose this condition mathematically, we put

$$\frac{\partial(\text{Im } \omega)}{\partial K} = 0$$

or in terms of the coefficients,

$$\begin{aligned} & 6G_1 x^5 + 5G_8 \psi x^4 + 4G_2 x^3 + 3G_7 \psi x^2 + 2G_3 x + G_6 \psi \\ &= - [a_7^2 (1+y)^2 + 4y]^{\frac{1}{2}} b_{31} [G_{12}(1+C_{11} + 3x^2) + 4G_{10} \psi x^3 \\ & \quad + 2G_9 \psi x] \end{aligned} \quad (29)$$

where $x = z^{\frac{1}{2}}$.

Combining equations (28) and (29) to eliminate

b_{31} , which is the only one of the three variables appearing in a simple enough form to be eliminated analytically, we obtain

$$\begin{aligned}
& 2G_1G_{10}\psi x^9 + 3G_1G_{12}x^8 + (4G_1G_9 + 2G_8G_{12})\psi x^7 + [5G_1G_{12} \cdot \\
& (1+C_{11}) + G_2G_{12}]x^6 + [4G_8G_{12}(1+C_{11}) + 2G_9G_2 - 2G_3G_{10} \\
& - 6G_1G_{11}]\psi x^5 + [3G_2G_{12}(1+C_{11}) - G_3G_{12}]x^4 + [2G_7G_{12} \cdot \\
& (1+C_{11}) - 2G_6G_{12} - 4G_4G_{10} + 4G_2G_{11}]\psi x^3 + [G_3G_{12}(1+C_{11}) \\
& - 3G_4G_{12}]x^2 + [2G_3G_{11} - 2G_9G_4]\psi x - G_4G_{12}(1+C_{11}) = 0 \quad (30)
\end{aligned}$$

Equation (30) is a relationship between z and y , which are essentially the wave number and the critical magnetic field. To evaluate the integrals, $p(r)$ is written in terms of the solution for $n(r)$ in the steady state taking only the lowest order term. That is, we take

$$n(r) = J_0(\gamma_0 r)$$

Similarly, E_{or} is replaced by its lowest order expression from the steady state.

Dividing the range of integration into ten equal parts we obtain the following numerical values for the integrals:

$$I_1 = -4.307$$

$$I_2 = 1.849$$

$$I_3 = - 9.570$$

$$I_4 = - 1.625$$

From the definition of ψ we write

$$\psi = \frac{(\mu_+ \mu_-)^{\frac{1}{2}} \bar{n} e (v_{z+} - v_{z-})}{c^2 K_o}$$

The orders of magnitude of the quantities involved are:

$$\mu_- \sim 10^5 \text{ cm}^2/\text{V-sec}$$

$$\mu_+ \sim 10^3 \text{ cm}^2/\text{V-sec}$$

$$\bar{n} \sim 10^{15}/\text{cm}^3$$

$$e \sim 10^{-10} \text{ esu}$$

$$v_z \sim 10^6 \text{ cm/sec}$$

$$c \sim 10^{10} \text{ cm/sec}$$

$$K_o \sim 1 / \text{cm} \quad (\text{Brown 1959})$$

which give for the order of magnitude of ψ

$$\psi \sim 10^{-5}$$

To accentuate the effect of the azimuthal field, equation (30) is plotted in figure 2 for parameter ψ equal to 0.001. Johnson's solution, corresponding to parameter ψ equal to zero, is also plotted in the same figure.

To obtain the graph of the electric field as a function of the critical magnetic field, we must eliminate the variable z between equations (28) and (29). This can be done only numerically by using the points of the graph in x and y . The relationship between the electric field variable b_{31} and the critical magnetic field is plotted in figure 3, again for parameter ψ equal to 0.001 and zero. This graph cannot be described in an algebraic form from equations (28) and (29).

The final quantity which we wish to plot as a function of the critical magnetic field is the frequency of rotation of the helix. We must return to equations (25) and solve for ω_1 which may be written

$$\omega_1 = \frac{\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$$

where A_1 , A_2 , C_1 , C_2 are given in (27) and where B_1 and B_2 are given by

$$B_1 = -A_2$$

$$B_2 = A_1$$

Therefore ω_1 may be written

$$\omega_1 = \frac{C_1 A_1 + C_2 A_2}{A_1^2 + A_2^2}$$

Substituting the values of A_1 , A_2 , C_1 , C_2 from equation (27), after much algebra, we obtain the following equation relating the real frequency ω_1 , the electric field variable b_{31} and the wave number variable x or z . This may be written as follows:

$$\begin{aligned} & b_{11} \left\{ D_{11}^2 a_7^2 + [1 + C_{11} + (1 + a_7^2)z]^2 + [2a_7^2 \frac{(2 + I_2) D_{11}}{(1+y)^{\frac{1}{2}}} z^{\frac{1}{2}} \right. \\ & \left. + 4(1 + C_{11} + z(1 + a_7^2)) \frac{yz^{\frac{1}{2}}}{(1+y)^{\frac{3}{2}}} \left(\frac{\mu_+}{\mu_-} + \frac{\mu_-}{\mu_+} + y - 1 \right) \right\} \psi / y^{\frac{1}{2}} \\ & = b_{31} \left\{ \left(z^{\frac{1}{2}} a_7 (1+y)^{\frac{1}{2}} [a_7^2 (1+y)^2 + 4y]^{\frac{1}{2}} [(1 + C_{11})^2 (1 + a_7^2)z] \right) \right. \\ & \left. + \frac{\psi}{y^{\frac{1}{2}}} [a_7(a_7^2 + (1+y)^2 + 4y)^{\frac{1}{2}} (1 + C_{11}) z^{\frac{1}{2}} (1+y)^{\frac{1}{2}} \left(\frac{2y z^{\frac{1}{2}}}{(1+y)^{\frac{1}{2}}} + \right. \right. \\ & \left. \left. z^{\frac{1}{2}} a_7^2 2(1+y)^{\frac{1}{2}} \right) + \left(z [a_7^2 (1+y)^2 + 4y]^{\frac{1}{2}} - (1 + C_{11}) \cdot \right. \right. \\ & \left. \left. [a_7^2 (1+y)^2 + 2y] \right) \left((1 + C_{11}) + (1 + a_7^2) z \right) \right\} + \left\{ a_7 D_{11} \cdot \right. \\ & \left[(1 + C_{11} + z) I_4 + 2(1 + C_{11}) + z I_2 + y D_{11} \right] - \frac{(2 + I_2)}{(1+y)^{\frac{1}{2}}} z^{\frac{1}{2}} a_7 \cdot \\ & \left. [D_{11} z^{\frac{1}{2}} (1+y)^{\frac{1}{2}}] \right\} + [a_7^2 (1+y)^2 + 4y]^{\frac{1}{2}} + a_7 D_{11} zy [(1 + C_{11})(1 - a_7^2) \\ & + z(1 + a_7^2)] + \psi \frac{1}{y^{\frac{1}{2}}} \left[\left\{ 2(1 + C_{11}) a_7 (1+y)^{\frac{1}{2}} z^{\frac{1}{2}} + I_2 \frac{1 + C_{11}}{(1+y)^{\frac{1}{2}}} a_7 z^{\frac{1}{2}} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + I_2 \frac{1+C_{11}}{(1+y)^{\frac{1}{2}}} a_7 z^{\frac{1}{2}} - C_{11} a_7 \frac{2+I_2}{(1+y)^{\frac{1}{2}}} z^{\frac{1}{2}} + (2+I_2)(1+y)^{\frac{1}{2}} a_7 z^{3/2} \\
& - D_{11} \frac{a_7 2y}{(1+y)^{\frac{1}{2}}} z^{\frac{1}{2}} - z^{\frac{1}{2}} \frac{(2+I_2)}{(1+y)^{\frac{1}{2}}} a_7 \left(\frac{\gamma_0}{K_0} \right)^2 \} \left\{ (1+C_{11}) + \right. \\
& \left. (1+a_7^2)y \right\} + \left\{ D_{11} a_7 + D_{11} z (1+y) a_7 - D_{11} a_7 \left(\frac{\gamma_0}{K_0} \right)^2 \right\} \cdot \\
& \left\{ \frac{2yz^{\frac{1}{2}}}{(1+y)^{\frac{1}{2}}} + 2(1+y)^{\frac{1}{2}} z^{\frac{1}{2}} a_7^2 \right\} - a_7 D_{11} \left\{ \frac{2z^{\frac{1}{2}}}{(1+y)^{\frac{1}{2}}} [a_7^2(1+y)^2 + 2y] \right. \\
& + 2C_{11} z^{\frac{1}{2}} y/(1+y)^{\frac{1}{2}} - 2 \left(\frac{\gamma_0}{K_0} \right)^2 z^{\frac{1}{2}} y/(1+y)^{\frac{1}{2}} + 2z^{3/2} [a_7^2(1+y)^2 + 2y]/ \\
& \left. (1+y)^{3/2} + 4z^{3/2} y^2/(1+y)^{3/2} + 2D_{11} z^{\frac{1}{2}} \frac{y}{(1+y)^{\frac{1}{2}}} \right\} - \left\{ \frac{(2+I_2)}{(1+y)^{\frac{1}{2}}} z^{\frac{1}{2}} a_7 \right\} \\
& \cdot \left\{ (1+C_{11}) + 2z + a_7^2 z (1+y) + C_{11} (1+a_7^2)y z - (1+C_{11}) \left(\frac{\gamma_0}{K_0} \right)^2 \right. \\
& \left. - z \left(\frac{\gamma_0}{K_0} \right)^2 (1+a_7^2) + z^2(1+a_7^2) \right\} \} \quad (31)
\end{aligned}$$

where b_{11} is defined as follows:

$$b_{11} = \frac{e \omega_1}{(kT_+ + kT_-)} \cdot \frac{\mu_+^i + \mu_-^i}{\mu_+^i \mu_-^i} \cdot \frac{1}{K_0^2}$$

Using the graphical results relating y and z we calculate b_{11} point by point from equation (31). The graph of b_{11} as a function of y is given in figure 4, again for ψ equal to 0.001 and zero.

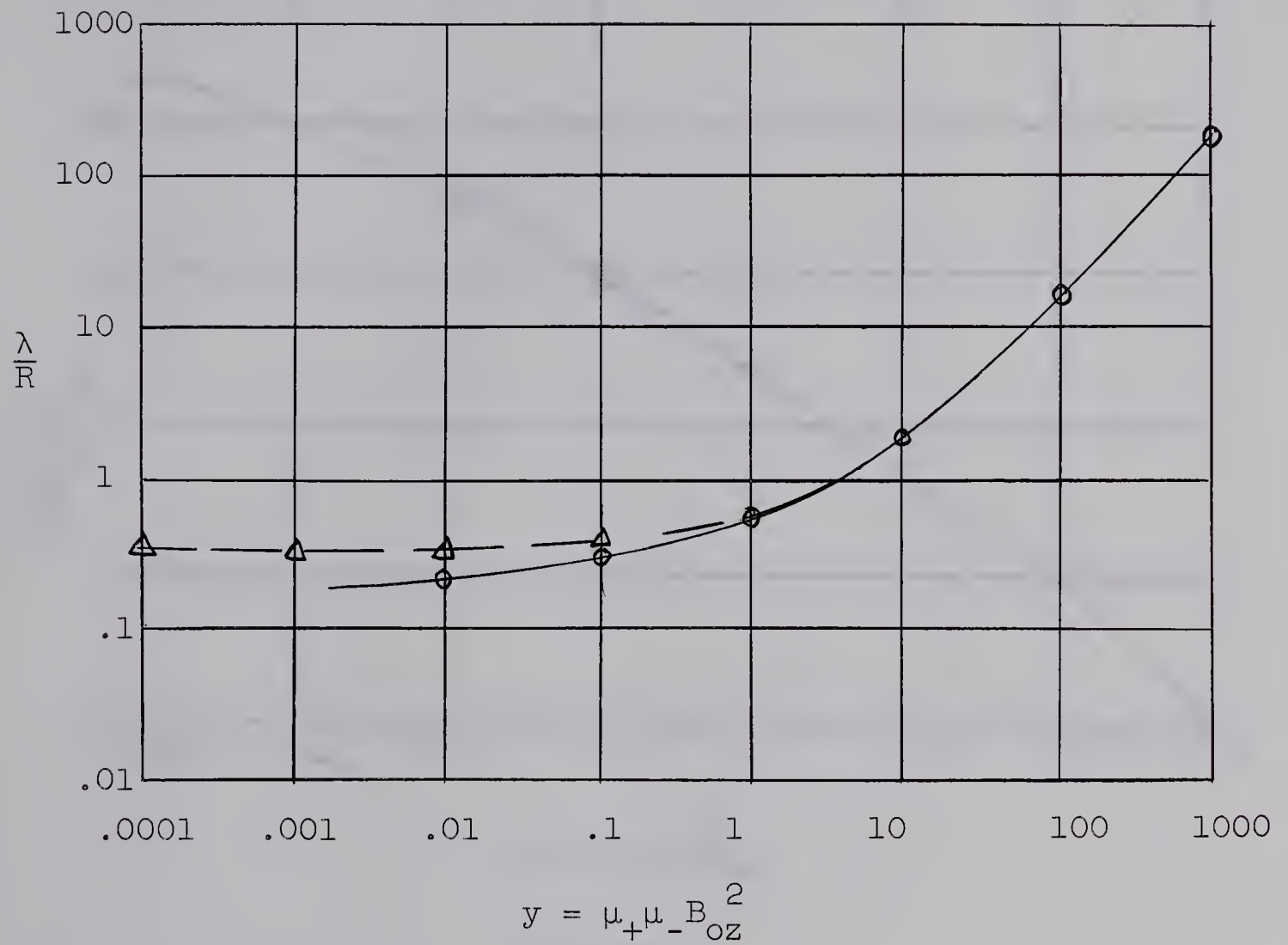


figure 2

Graph of the wavelength λ of the helix as a function of the critical longitudinal magnetic field for parameter ψ equal to .001 (solid curve) and zero (Johnson's solution, dashed curve)

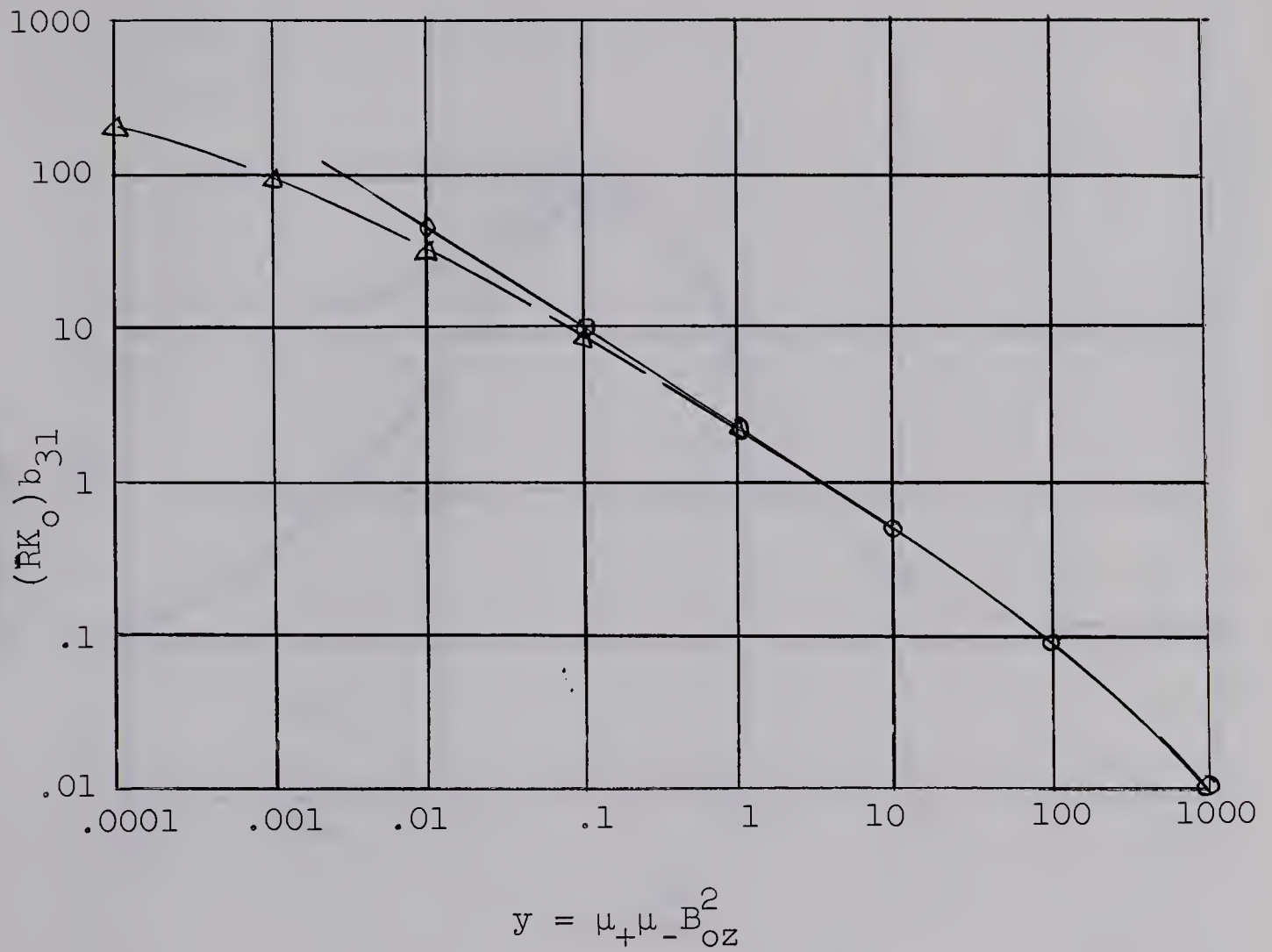


figure 3

Graph of the electric field variable $b_{31} = \frac{E_{OZ}}{K_O(kT_+ + kT_-)}$ as a function of the critical magnetic field for parameter ψ equal to .001 (solid curve) and zero (Johnson's solution, dashed curve)

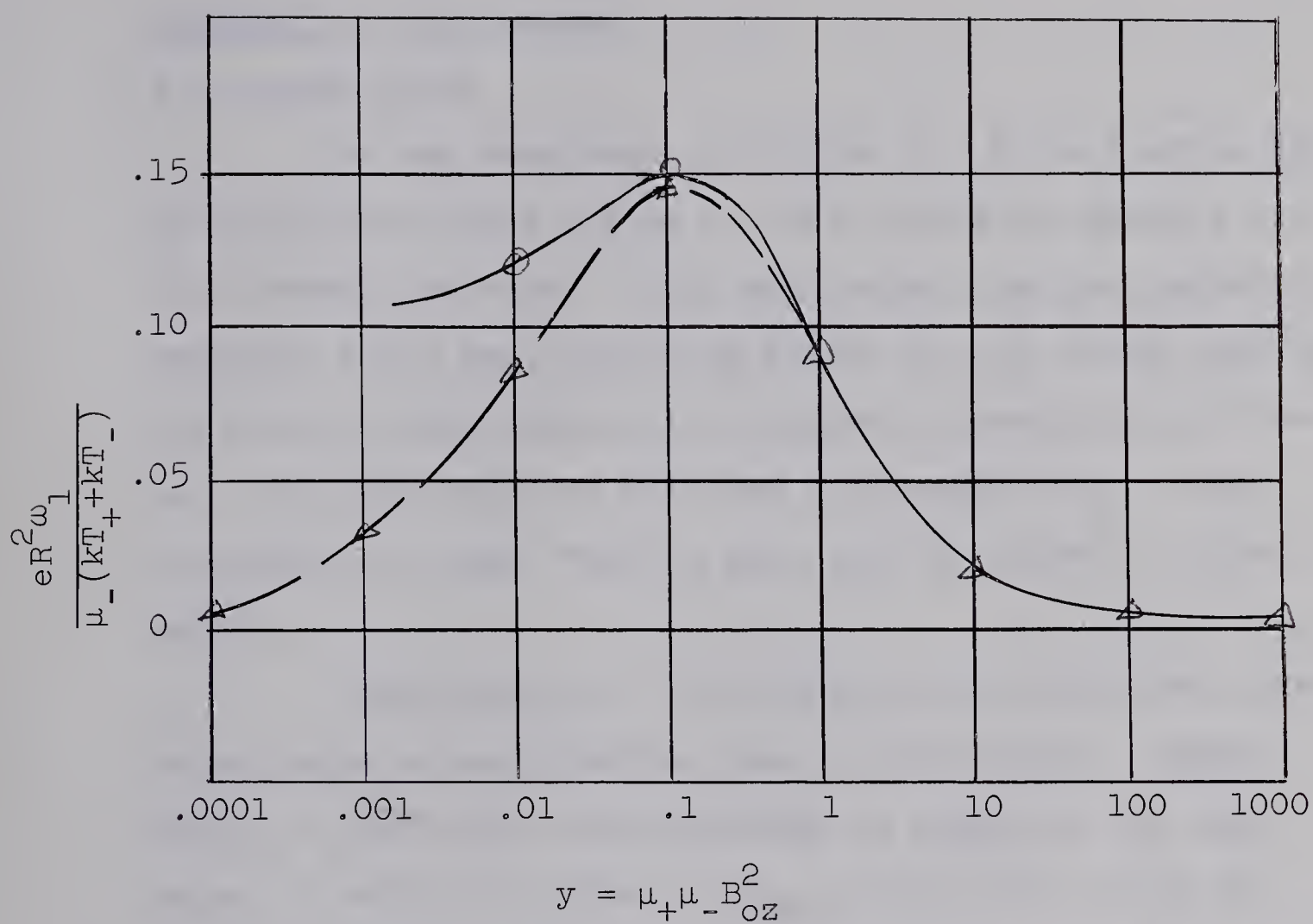


figure 4

Graph of the frequency of the helix as a function of the critical magnetic field for ψ equal to 0.001 (solid curve) and ψ equal to zero (Johnson's solution, dashed curve)

Chapter 4: Conclusions

4.1 Steady State

As was mentioned in Chapter 2, it is easy to explain qualitatively figure 1, which shows the density profile across the tube. It is well known that the azimuthal magnetic field has a pinching effect on the column and its inclusion indeed results in a higher concentration of ions and electrons near the axis and a correspondingly lower concentration away from the axis than in Johnson's calculation.

Verification of the effect of the additional term using existing experimental data is impossible. Experiments to date have been performed in ranges of the variables in which the effect of B_{ϕ} is too small to be observed. That is, they have been performed using low average densities and low currents.

4.2 Helical mode

The graphs of figures 2, 3 and 4 show the deviation from Johnson's calculation for a value of the dimensionless azimuthal field parameter ψ equal to 0.001. Although an experiment in this range is possible, it would have to be one of unusually high current, strong electric field and high average density. No experiment has been performed to our knowledge, which was designed to isolate

the effect of the azimuthal field. The usual experiments, investigating the helical mode, are performed for values of y greater than those where Johnson's theory and the present modified theory disagree.

A representative example of such experiments would be one described in a recent paper by M. Sato. This paper gives a single point on the graph of wavelength versus critical field (figure 2). At the minimum of light intensity from the column, which is assumed to occur at the critical field, a helix of wavelength $\lambda = 4 R$ is observed when the critical magnetic field parameter $y = 10^{-6}$. This agrees with the asymptotic behaviour of Johnson's graph. However, under the conditions of the experiment, $\psi = 10^{-7}$ and the deviation due to the induced field in this case is important only for $y < 10^{-6}$. Furthermore, the observation has no more than order-of-magnitude precision, and since the deviation between Johnson's calculation and the present modification is small, the only conclusion to be drawn from this experimental point is that it agrees with both calculations within the limits of its precision.

It is interesting to study the behaviour of Johnson's calculation in the limit that the longitudinal magnetic field goes to zero. In Johnson's calculation, putting the longitudinal magnetic field to zero means there is no longer any magnetic field in any direction. The problem reduces to a simple electric discharge in a longitudin-

al electric field with a sink at the boundary. The problem now has cylindrical symmetry and consequently no helix should exist. If a helical state were imposed on the system, it would quickly decay due to the frictional force retarding the azimuthal motion. There is no force which could sustain the azimuthal motion. The frequency ω and the wavelength λ may be investigated in this limit and it is found that ω tends to zero and λ tends to a finite value. In other words, Johnson's calculation admits a stationary helix as solution for $B_{oz} = 0$.

It must be noted that we have included the induced field as a perturbation to Johnson's solution. Consequently, the range of validity of the modified results is limited to cases of small deviations from Johnson's results. The modification is valid if the deviations in the wavelength, electric field, and rotational frequency are much smaller than the values of the wavelength, electric field, and frequency respectively.

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